MATH 5061 Problem Set 2[1]
Due date: Mar 4, 2020

Problems: (Please either type up your assignment or scan a copy of your written assignment into ONE PDF file and send it to me by email on/before the due date. Please remember to write down your name and SID.)

1. A vector field X is said to be *complete* if its flow φ_t is defined for all $t \in \mathbb{R}$.
   (a) Prove that any vector field on a compact manifold is complete. *(Hint: By Whitney embedding theorem, you can assume that the compact manifold is embedded in some $\mathbb{R}^N$ for some N.)*
   (b) Give an example of an incomplete vector field on $\mathbb{R}$.

2. Give an alternative proof that $\mathcal{L}_X Y = [X, Y]$ along the following lines.
   (a) First consider the case when $X(P) \neq 0$ and choose coordinates $x^1, \cdots, x^n$ near P in which $X = \frac{\partial}{\partial x^1}$. Explicitly determine the flow φ_t in these coordinates, write Y in coordinates, and check directly that
   $$[X, Y] = \lim_{t \to 0} \frac{(\varphi_{-t})_*(Y(\varphi_t(P)) - Y(P)}{t}.$$
   (b) Explain why it is sufficient to check that $\mathcal{L}_X Y = [X, Y]$ in a coordinate system. Give a continuity argument to show that $\mathcal{L}_X Y = [X, Y]$ even at points where $X = 0$.

3. Let V be an m-dimensional real vector space. An element $\sigma \in \wedge^k V$ is called *simple* if there exist $v_1, \cdots, v_k \in V$ such that $\sigma = v_1 \wedge \cdots \wedge v_k$.
   (a) Show that every element of $\wedge^{m-1} V$ is simple. *(Hint: Fix $\sigma \in \wedge^{m-1} V$, consider the map $v \mapsto \sigma \wedge v$.)*
   (b) Give an example of a non-simple element in $\wedge^2 \mathbb{R}^4$.
   (c) Let W be a k-dimensional subspace of V. Given a basis $e_1, \cdots, e_k$ for W, consider the element $e_1 \wedge \cdots \wedge e_k$. Examine what happens when one chooses a different basis, and construct a $1-1$ map from the collection of all k dimensional subspaces of V onto the subset of simple elements of the projective space (space of lines through the origin) on the vector space $\wedge^k V$.

4. Let V^m be a vector space with a positive definite scalar product g. We extend the scalar product to $\wedge^k V$ be requiring that
   $$g(v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k) = \det(g(v_i, w_j))$$
   on simple k-vectors. Note that this implies that if $e_1, \cdots, e_m$ is an orthonormal basis for (V, g), then $e_{i_1} \wedge \cdots \wedge e_{i_k}$, $1 \leq i_1 < i_2 < \cdots < i_k \leq m$, is an orthonormal basis for $\wedge^k V$.
   (a) Show that the pairing $\wedge^k V \times \wedge^{m-k} V \to \wedge^m V$ given by the wedge product $(\alpha, \beta) \mapsto \alpha \wedge \beta$ is non-degenerate.
   (b) It follows that if we choose $*1 \in \wedge^m V$ to be one of the two unit m-vectors (this amounts to choosing an orientation on V), then we may define a linear transformation $* : \wedge^k V \to \wedge^{m-k} V$ by requiring
   $$\alpha \wedge *\beta = g(\alpha, \beta) * 1.$$
   Show that $*$ is a linear isomorphism.
   (c) Show that $*(v_1 \wedge \cdots \wedge v_k)$ is a simple vector which represents the $(m-k)$-plane which is the orthogonal complement of the k-plane spanned by $v_1, \cdots, v_k$.
   (d) Show that $*^2 = (-1)^{k(m-k)}$.

5. (a) Show that if $m = 2$, then $* : V \to V$ may be interpreted as rotation by $90°$ in the counterclockwise direction. Thus, $*$ defines a complex structure on V.
   (b) Show that for $m = 4$, $k = 2$, we have $*^2 = 1$, and $*$ defines an orthogonal direct sum decomposition of $\wedge^2 V$ into the $+1$ and -1 eigenspaces. A 2-vector α with $*\alpha = \alpha$ (resp. $*\alpha = -\alpha$) is called a *self-dual* (resp. *anti-self-dual*) 2-vector.

[1]Last revised on February 18, 2020